Name solutions

EE 311 Final Exam Fall 2011

December 13, 2010

Closed Text and Notes, No calculators

1) Be sure you have 14 pages and the additional pages of equations.

2) Write only on the question sheets. Show all your work. If you need more room for a particular problem, use the reverse side of the same page.

3) Write neatly, if your writing is illegible then print.

5) This exam is worth 150 points.

(9 pts) 1. Given the vectors $\mathbf{A} = 3\hat{\mathbf{a}}_x + 4\,\hat{\mathbf{a}}_z$ and $\mathbf{B} = 2\hat{\mathbf{a}}_x + 3\,\hat{\mathbf{a}}_y + 2\hat{\mathbf{a}}_z$

(3 pts) a) Find a unit vector in the direction of A

$$\hat{a}_A = \frac{\vec{A}}{A} = \frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}} = \frac{3\hat{a}_x + 4\hat{a}_z}{\sqrt{9+16}}$$

$$\hat{a}_A = \frac{3}{5}\hat{a}_x + \frac{4}{5}\hat{a}_z$$

(3 pts) b) Find $\mathbf{A}\bullet\mathbf{B}$

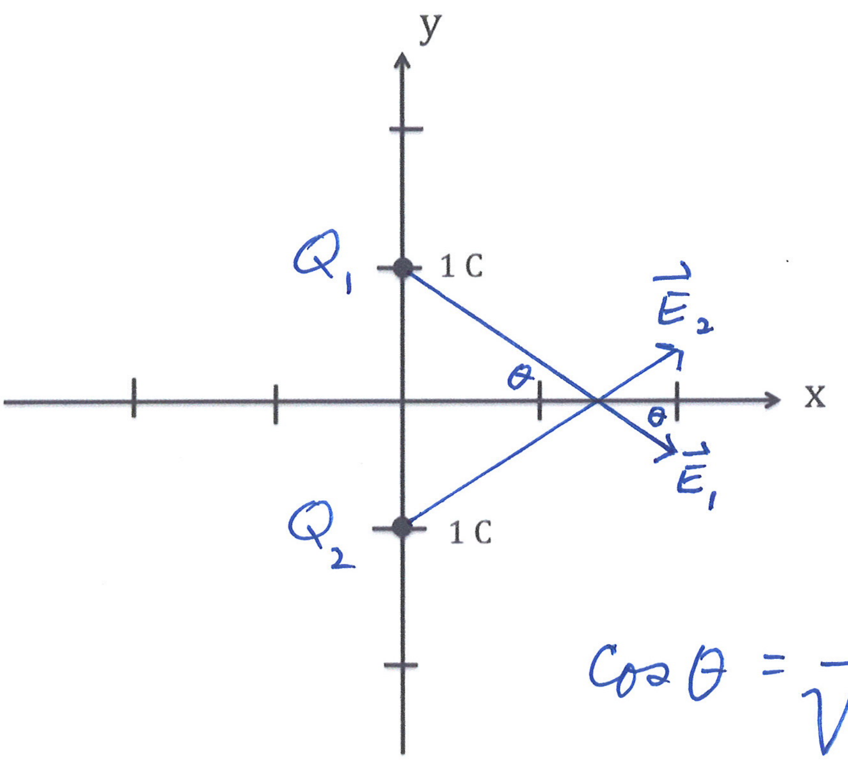
$$\vec{A}\cdot\vec{B} = (3\hat{a}_x + 4\hat{a}_z)\cdot(2\hat{a}_x + 3\hat{a}_y + 2\hat{a}_z)$$
$$= (3)(2) + (0)(3) + (4)(2)$$
$$= 14$$

(3 pts) c) Find $\mathbf{A}\times\mathbf{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 3 & 0 & 4 \\ 2 & 3 & 2 \end{vmatrix} = \begin{aligned} &\hat{a}_x(0-12) \\ &-\hat{a}_y(6-8) \\ &+\hat{a}_z(9-0) \end{aligned}$$

$$= -12\hat{a}_x + 2\hat{a}_y + 9\hat{a}_z$$

(10 pts) 2. There is a charge of 1 C at (0, 1m, 0) and at (0, -1m, 0) as shown. Find $\mathbf{E}(x,0,0)$, the electric field on the x-axis.



On the x-axis, the $\hat{a}_y$ components of the $\vec{E}$ field from Q_1 and Q_2 will cancel and the $\hat{a}_x$ components will add.

$$\cos\theta = \frac{x}{\sqrt{1+x^2}}$$

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{1}{(1+x^2)}$$

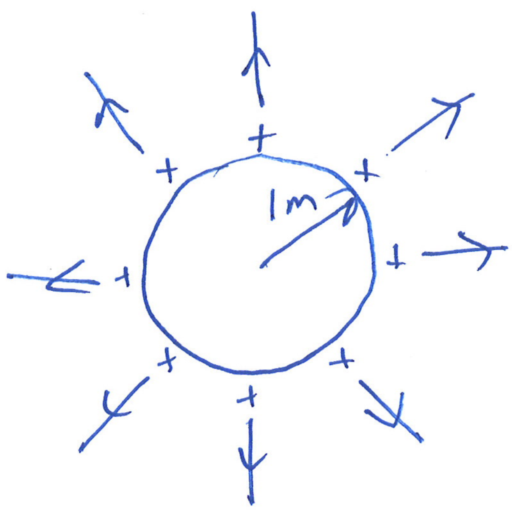
$$\vec{E}(x,0,0) = 2\cos\theta\, E_1\, \hat{a}_x$$

$$= 2\,\frac{x}{\sqrt{1+x^2}}\,\frac{1}{4\pi\epsilon_0}\,\frac{1}{1+x^2}$$

$$= \frac{x}{2\pi\epsilon_0(1+x^2)^{3/2}}\,\hat{a}_x$$

(10 pts) 3. A conducting spherical shell of radius 1 m has a charge of $\frac{10^{-9}}{9}$C on it. If $V(\infty)=0$, find V(r) everywhere. $\varepsilon_o = \frac{10^{-9}}{36\pi}\frac{F}{m}$

for $r < 1m$ $\vec{E} = 0$



for $r > 1m$ the $\vec{D}$ field has sperical symmetry

$$\oint \vec{D} \cdot \vec{ds} = Q_{encl}$$

$$D\, 4\pi r^2 = \frac{10^{-9}}{9} C$$

$$\vec{D} = \frac{\left(\frac{10^{-9}}{9}\right)}{4\pi r^2}\hat{a}_r \quad \text{for } r > 1m$$

$$\vec{E} = \frac{\left(\frac{10^{-9}}{9}\right)C}{4\pi \epsilon_o r^2}\hat{a}_r = \frac{\left(\frac{10^{-9}}{9}\right)C}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)r^2}\hat{a}_r$$

$$= \frac{1}{r^2}\hat{a}_r \quad \text{for } r > 1m$$

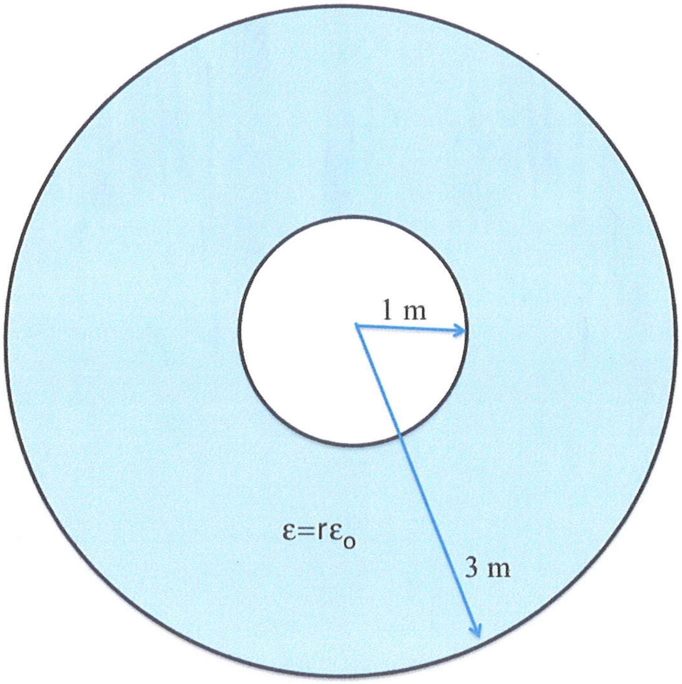
$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl} = -\int_{\infty}^{r} \frac{1}{r^2} dr = \frac{1}{r}\Big|_{\infty}^{r} = \frac{1}{r}$$

for $r > 1m$

$$V(r) = \frac{1}{r} \quad \text{for } r \geq 1m$$

$$= 1V \quad \text{for } r \leq 1m$$

(15 pts) 4. A capacitor consists of two co-centric hollow metal spheres. The radius of the inner sphere is 1m and the radius of the outer sphere is 3m. The permittivity of the dielectric between the spheres varies with position as $\varepsilon = r\varepsilon_o$. Determine the capacitance.



place a charge of +Q on the inner sphere and −Q on the outer sphere.

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r, \quad 1m < r < 3m$$

$$\vec{E} = \frac{Q}{4\pi \epsilon_o r^3}\hat{a}_r, \quad 1m < r < 3m$$

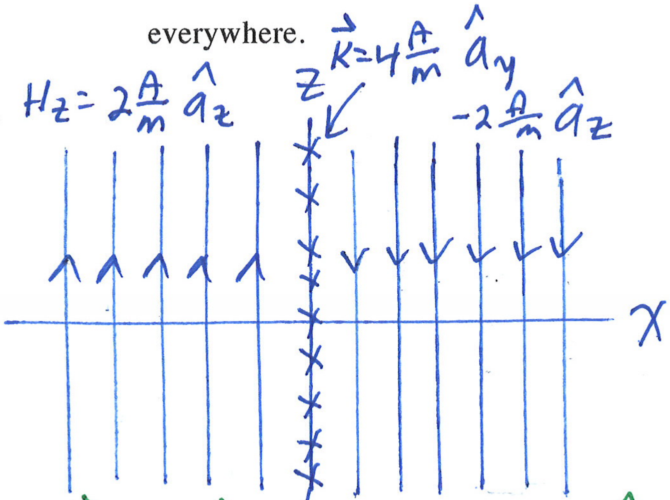
$$V = -\int_{3m}^{1m} \vec{E}\cdot d\vec{l} = -\int_{3m}^{1m} \frac{Q}{4\pi\epsilon_o r^3}\,dr = -\frac{Q}{4\pi\epsilon_o}\int_{3m}^{1m}\frac{dr}{r^3}$$

$$= \frac{Q}{8\pi\epsilon_o}\left.\frac{1}{r^2}\right|_3^1 = \frac{Q}{8\pi\epsilon_o}\left(\frac{1}{1}-\frac{1}{9}\right)$$

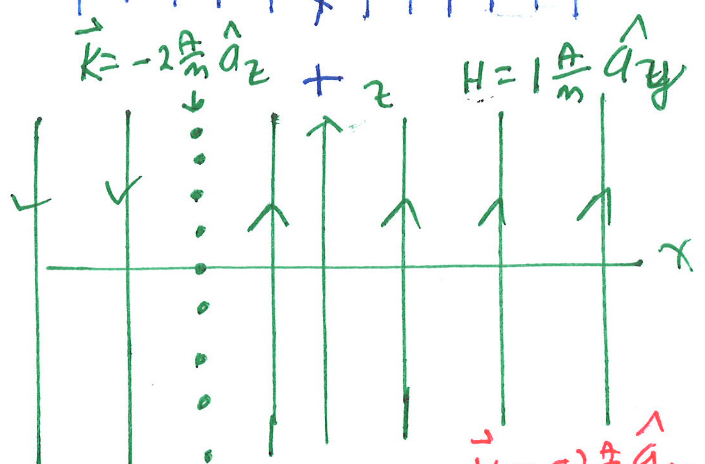
$$= \frac{Q}{8\pi\epsilon_o}\,\frac{8}{9} = \frac{Q}{9\pi\epsilon_o}$$

$$C = \frac{Q}{V} = \frac{Q}{\left(\frac{Q}{9\pi\epsilon_o}\right)} = 9\pi\epsilon_o$$

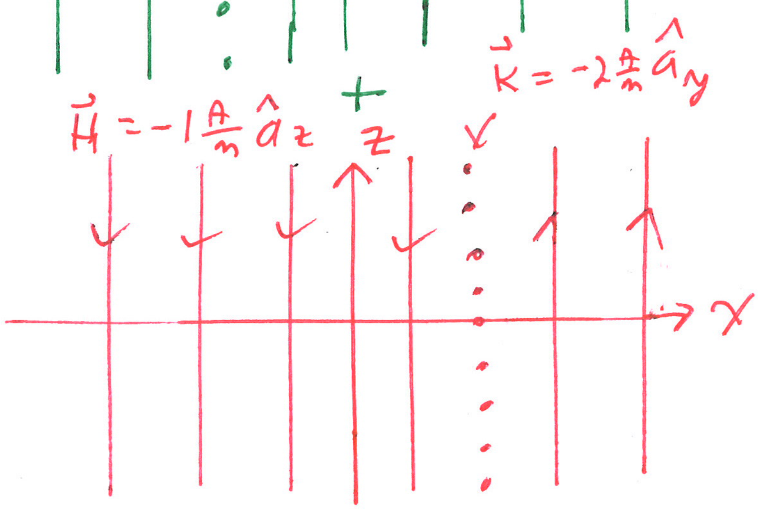
(9 pts) 5. A sheet current density of $\mathbf{K}=4\frac{\mathrm{A}}{\mathrm{m}}\,\hat{\mathbf{a}}_y$ is flowing on the x = 0 plane, $\mathbf{K}=-2\,\frac{\mathrm{A}}{\mathrm{m}}\,\hat{\mathbf{a}}_y$ on the x = -1m plane and $\mathbf{K}=-2\,\frac{\mathrm{A}}{\mathrm{m}}\,\hat{\mathbf{a}}_y$ on the x = 1 m plane. Determine the magnetic field intensity everywhere.



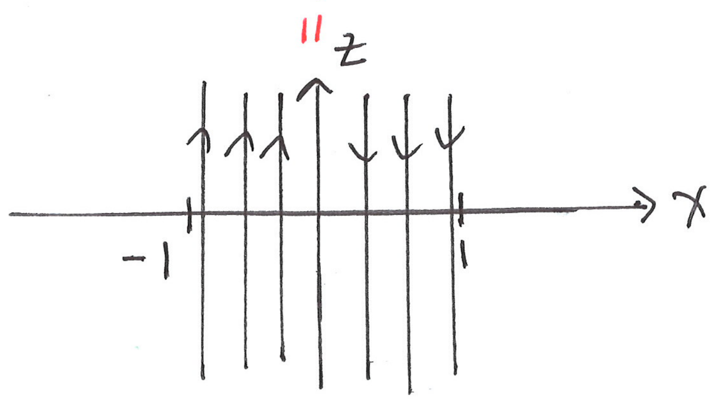
$$\vec{H} = 2\frac{A}{m}\hat{a}_z \quad x<0$$
$$-2\frac{A}{m}\hat{a}_z \quad x>0$$



$$\vec{H} = -1\frac{A}{m}\hat{a}_z \quad x<-1m$$
$$= 1\frac{A}{m}\hat{a}_z \quad x>-1m$$



$$\vec{H} = -1\frac{A}{m}\hat{a}_z \quad x<1m$$
$$= 1\frac{A}{m}\hat{a}_z \quad x>1m$$



$$\vec{H} = 0 \quad x<-1m$$
$$= 2\frac{A}{m}\hat{a}_z \quad -1m<x<0$$
$$= -2\frac{A}{m}\hat{a}_z \quad 0<x<1m$$
$$= 0 \quad x>1m$$

(6 pts) 6. A current of 10 A is flowing in the +z direction along the z axis . The value of $\oint \mathbf{B} \bullet d\mathbf{S}$ over a sphere of radius 1 m centered at the origin is,

0

(5 pts) 7. A magnetic material has $2x10^{29}$ atoms/m^3, each with a magnetic dipole moment of $2x10^{-27}$ $\mathbf{a}_z$ Am^2. What is the value of **M** for this material, with appropriate units?

$$\vec{M} = \frac{\text{dipole moment}}{\text{unit volume}} = \left(2 \times 10^{29} \frac{\text{atoms}}{m^3}\right)\left(2 \times 10^{-27} \hat{a}_z \frac{Am^2}{\text{atom}}\right)$$

$$= 4 \times 10^{2} \hat{a}_z \frac{A}{m}$$

(5 pts) 8. Two parallel wires carry currents in the same direction. The force experienced by one due to the other is

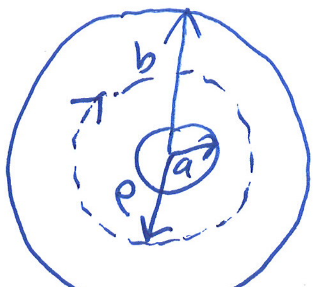
A) parallel to the wires

B) perpendicular to the wires and attractive

C) perpendicular to the wires and repulsive

D) zero.

(15 pts) 9. Find the inductance per unit length for a coaxial cable with inner conductor of radius a and outer conductor of radius b. The dielectric between the conductors has permeability μ_o and permittivity of $2\varepsilon_o$.

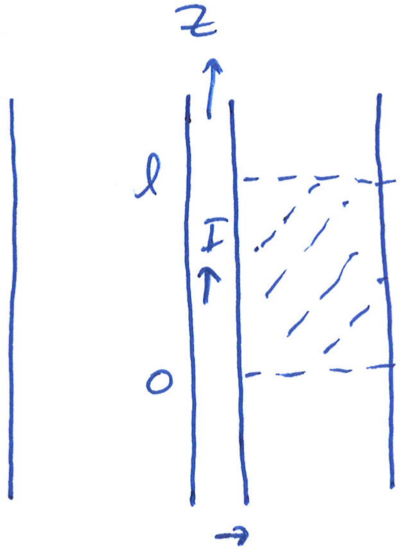


First use Ampere's Law to find the magnetic field in the coax. Align the coax on the z axis with a current I flowing in the $+\hat{a}_z$ direction. This system has cylindrical symmetry

$$\oint \vec{H}\cdot d\vec{l} = I \quad \text{for } a<\rho<b$$

$$H\,2\pi\rho = I \Rightarrow \vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi$$

for $a<\rho<b$



Find the amount of magnetic flux through the plane shown from $\rho = a$ to b and $z=0$ to l

$$\phi = \int_0^l \int_a^b \vec{B}\cdot d\vec{S} = \int_0^l \int_a^b \frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi \cdot d\rho\, dz\, \hat{a}_\phi$$

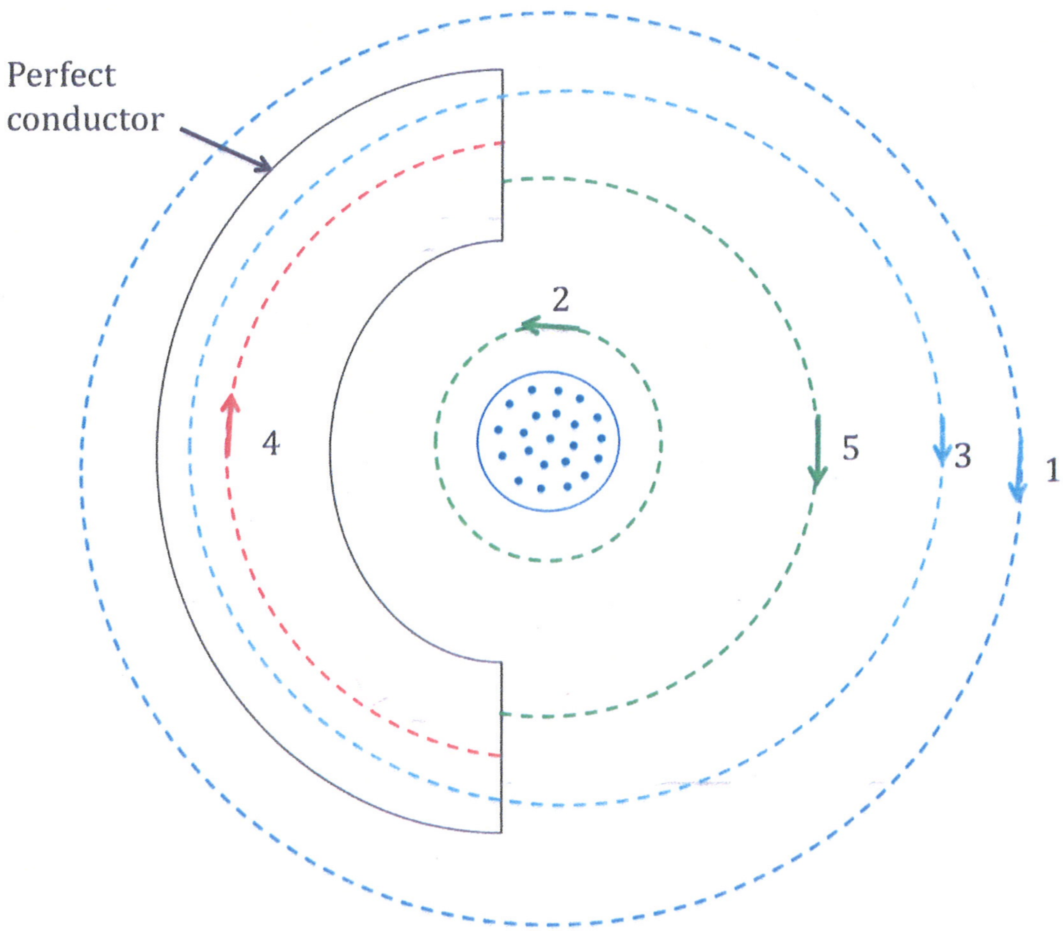
$$\phi = \frac{\mu_0 I}{2\pi}\int_0^l \int_a^b \frac{1}{\rho}\, d\rho\, dz = \frac{\mu_0 I l}{2\pi}\int_a^b \frac{d\rho}{\rho}$$

$$\phi = \frac{\mu_0 I l}{2\pi}\ln\rho \Big|_a^b = \frac{\mu_0 I l}{2\pi}(\ln b - \ln a) = \frac{\mu_0 I l}{2\pi}\ln\frac{b}{a}$$

$$L = \frac{\phi}{I} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

(15 pts) 10. Shown is a half ring that is a perfect conductor in the plane of the page. Shown is a solenoid that is perpendicular to the page and through the plane of the page. The dots in the center of the solenoid indicate the direction the magnetic flux is increasing due to an increasing current in the solenoid resulting in a rate of change of magnetic flux out-of-the page of $\frac{d\phi}{dt} = 10\,V$. The dashed lines indicate paths. For the paths shown find the following,



$\oint_1 \mathbf{E} \bullet \mathbf{dl} =$ 10 V

$\oint_2 \mathbf{E} \bullet \mathbf{dl} =$ - 10V

$\oint_3 \mathbf{E} \bullet \mathbf{dl} =$ 10 V

$\int_4 \mathbf{E} \bullet \mathbf{dl} =$ 0

$\int_5 \mathbf{E} \bullet \mathbf{dl} =$ 10V

(17 pts) 11. A transverse electromagnetic uniform plane wave is propagating in a non-magnetic dielectric. The electric field associated with this wave is given by,

$$\mathbf{E} = 10\frac{V}{m}\cos\left(\frac{2\pi}{10}x + 2\pi 10^7 t\right)\hat{\mathbf{a}}_y$$

(3 pts) a) What is the wavelength?

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{10} \qquad \lambda = 10m$$

(3 pts) b) What is the period?

$$\omega = 2\pi f = 2\pi 10^{+7} s^{-1} \Longrightarrow f = 10^7 s^{-1}$$

$$T = \frac{1}{f} = 10^{-7} s$$

(3 pts) c) what is the velocity including direction.

The wave is traveling in the $-\hat{a}_x$ direction

$$\vec{u} = -\frac{\omega}{\beta}\hat{a}_x = -\frac{2\times10^7 s^{-1}}{\left(\frac{2\pi}{10} m^{-1}\right)}\hat{a}_x = -10^8 \frac{m}{s}\hat{a}_x$$

(8 pts) Determine the magnetic field intensity. Note $\frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3\text{x}10^8 \frac{m}{s}$ and $\sqrt{\frac{\mu_o}{\varepsilon_o}} = 377\,\Omega$

$$\vec{H} = -\frac{10\frac{V}{m}}{\eta}\cos\left(\frac{2\pi}{10}x + 2\pi 10^7 t\right)\hat{a}_z$$

$$u = 10^8 \frac{m}{s} = \frac{1}{\sqrt{\mu_o \epsilon_r \epsilon_o}} = \frac{1}{\sqrt{\epsilon_r}}\frac{1}{\sqrt{\mu_o \epsilon_o}} = \frac{1}{\sqrt{\epsilon_r}} 3\times10^8 \frac{m}{s}$$

$$\frac{1}{\sqrt{\epsilon_r}} = \frac{1}{3} \qquad \eta = \sqrt{\frac{\mu_o}{\epsilon_r \epsilon_o}} = \frac{1}{\sqrt{\epsilon_r}}\sqrt{\frac{\mu_o}{\epsilon_o}} = \frac{1}{3}(377\Omega) = 125.6\Omega$$

$$\vec{H} = -\frac{10}{125.6}\frac{A}{m}\cos\left(\frac{2\pi}{10}x + 2\pi 10^7 t\right)\hat{a}_z$$

(15 pts) 12. The region z < 0 is free space and the region z > 0 is a lossly dielectric. The electric field is

$$\mathbf{E} = 15\frac{V}{m}\cos(\frac{2\pi}{10}z - 6\pi 10^7 t)\hat{\mathbf{a}}_x - 10\frac{V}{m}\cos(\frac{2\pi}{10}z + 6\pi 10^7 t)\hat{\mathbf{a}}_x \quad \text{for } z < 0$$

and

$$\mathbf{E} = 5\frac{V}{m}e^{-5z}\cos(\frac{2\pi}{10}z - 3\pi 10^7 t)\hat{\mathbf{a}}_x \quad \text{for } z > 0$$

(5 pts) What is the reflection coefficient?

$$\Gamma = \frac{-10}{15} = -\frac{2}{3}$$

(5 pts) What is the transmission coefficient?

$$\tau = \frac{5}{15} = \frac{1}{3}$$

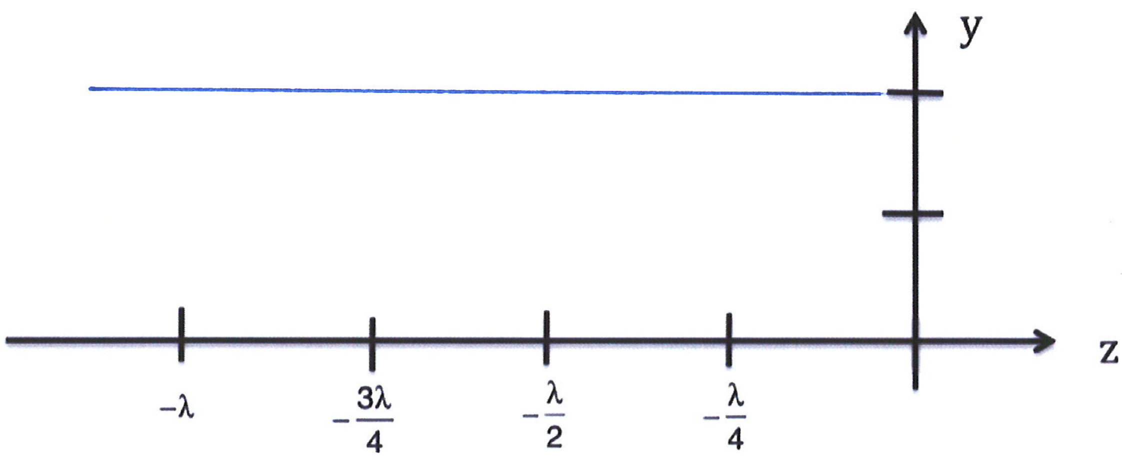
(5 pts) What is the skin depth in the lossy dielectric?

$$-5z = -1$$

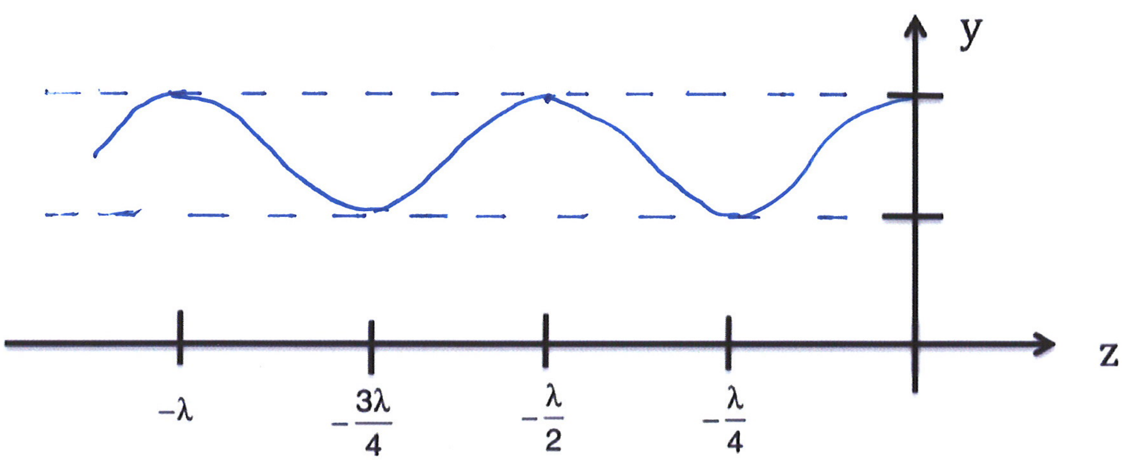
$$z = \frac{1}{5}\,m = \delta$$

(9 pts) 13. Draw the amplitude of the electric field with a standing wave ratio of

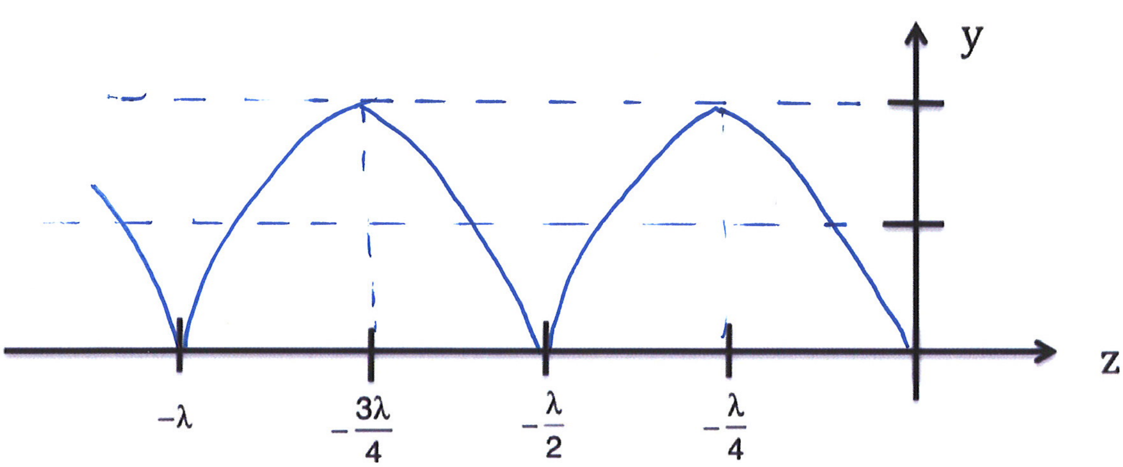
(3 pts) a) S = 1



(3 pts) b) S = 2



(3 pts) c) S = ∞



(10 pts) 14. For the following electric field,

$$\mathbf{E} = 15\frac{\text{V}}{\text{m}}\cos\left(\frac{2\pi}{10}\frac{x}{\sqrt{2}} + \frac{2\pi}{10}\frac{y}{\sqrt{2}} - 6\pi 10^7 t\right)\hat{\mathbf{a}}_z$$

(5 pts) What is the wavelength?

$$\vec{k} = k_x \hat{a}_x + k_y \hat{a}_y = \frac{2\pi}{10\sqrt{2}}\hat{a}_x + \frac{2\pi}{10\sqrt{2}}\hat{a}_y$$

$$k = \frac{2\pi}{\lambda} = \sqrt{\vec{k}\cdot\vec{k}} = \sqrt{\left(\frac{2\pi}{10\sqrt{2}}\right)^2 + \left(\frac{2\pi}{10\sqrt{2}}\right)^2} =$$

$$= \sqrt{2\left(\frac{2\pi}{10\sqrt{2}}\right)^2} = \sqrt{2}\,\frac{2\pi}{10\sqrt{2}} = \frac{2\pi}{10}$$

$$\lambda = 10\,m$$

(5 pts) Find a unit vector in the direction of propagation.

$$\hat{a}_k = \frac{\vec{k}}{k} = \frac{\frac{2\pi}{10\sqrt{2}}\hat{a}_x + \frac{2\pi}{10\sqrt{2}}\hat{a}_y}{\left(2\pi/10\right)}$$

$$\hat{a}_k = \frac{1}{\sqrt{2}}\hat{a}_x + \frac{1}{\sqrt{2}}\hat{a}_y$$